

Ian G Moss and James P Norman

Department of Physics, University of Newcastle Upon Tyne, NE1 7RU U.K.

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Abstract

Quasinormal mode spectra for gravitational perturbations of black holes in four dimensional de Sitter and anti-de Sitter space are investigated. The de Sitter case lends itself to approximation by a Pöschl-Teller potential. The anti-de Sitter case is relevant to the ADS-CFT correspondence in superstring theory. The ADS-CFT correspondence suggests a preferred set of boundary conditions.

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I. INTRODUCTION

The spectrum of gravitational waves emitted by a black hole which has been disturbed is dominated, over intermediate timescales, by characteristic frequencies and decay timescales. These can be accurately described in terms of the discrete set of quasinormal modes. The frequencies of these modes can be determined numerically these days to high accuracy up to large order [1].

A new application of the quasinormal mode spectrum has arisen recently from superstring theory. There is a suggestion, known as the ADS-CFT correspondence, that string theory in anti-de Sitter space (usually with extra internal dimensions) is equivalent to conformal field theory in one less dimension [3,4]. A black hole, which has a characteristic temperature fixed by the Hawking effect, should correspond to CFT at finite temperature and the quasinormal modes would describe non-equilibrium effects [5].

The temperature of the black hole is fixed by the surface gravity of the event horizon κ_1 . The anti-de Sitter space also has a characteristic length scale a which is conventionally referred to as the anti-de Sitter radius. In the non-rotating case, there are two black hole solutions for a given value of the surface gravity [6], a larger one with positive and smaller one with negative specific heat. Therefore the metric is determined by a , κ_1 and the choice of the larger or the smaller black hole.

Previous work on the quasinormal mode spectrum in anti-de Sitter space has concentrated on the wave equation for scalar fields. Chan et al. [7,8] studied the scalar field problem prior to the interest in the ADS-CFT correspondence. Horowitz et al. [9,10] examined the scalar spectrum in four and five dimensions. They found simple scaling behaviour of the spectrum in both limits $r_1 \gg a$ and $r_1 \ll a$, the former explained by an approximate symmetry of the metric in that limit.

Gravitational perturbations of the BTZ black hole in three dimensions and the anti-de Sitter black hole in four dimensions have been studied by Cardoso et al. [11,12]. They make use of the fact that the gravitational perturbation equation is separable in four dimensions [13–15]. This fact was also used in a preliminary report on our results [16]. The main difference between our work and the work of Cardoso et al. is in the choice of boundary conditions on the axial and polar metric perturbations.

There is a symmetry which relates the axial and polar metric perturbations, whose existence can be explained by the fact that the covariant form of perturbation analysis based on the Weyl tensor is governed by a single equation [18]. We argue that the thermal fluctuations in the energy of the conformal field theory are determined by a single set of modes for which the axial perturbations have dirichlet boundary conditions and the duality between axial and polar perturbations is preserved. We calculate the quasinormal mode frequencies numerically. For fluctuations in other components of the stress energy tensor there is a one parameter family of boundary conditions and the modes depend weakly on this parameter.

We have also investigated the quasinormal mode spectrum for black holes in de Sitter space. An interesting feature of this problem is that the asymptotic behaviour of the modes is under better control than the corresponding problem in flat space [1]. The first few modes were evaluated previously in ref. [13]. Using methods based on continued fractions now gives us many thousands of modes. A Pöschl-Teller approximation for the potential works

extremely well in this example [16].

II. DE SITTER SPACE

The spherically symmetric de Sitter spacetime which contains a black hole has two horizons, one at the event horizon of the black hole and a cosmological horizon further out. In standard coordinates, the metric for a black hole of mass M and for a cosmological constant Λ can be written

$$ds^2 = -\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1)$$

where

$$\Delta = r^2 - 2Mr - \frac{1}{3}\Lambda r^4. \quad (2)$$

The event horizon lies at the smallest root r_1 of Δ , where the surface gravity $\kappa_1 = r^{-2}\Delta'/2$. We can invert these relations to give

$$M = \frac{1}{3}r_1(\kappa_1 r_1 + 1) \quad (3)$$

$$\Lambda = r_1^{-2}(1 - 2\kappa_1 r_1), \quad (4)$$

showing, incidentally, that $\kappa_1 r_1 < 1/2$ in de Sitter space.

The gravitational wave equations were separated originally for charged black holes [13], then for uncharged [14] and finally for rotating black holes [15]. The metric perturbations can be classified by their axial or polar symmetry under spatial inversions. The transformation theory relating the two types is described in the next section.

The axial metric perturbations with time dependence $e^{i\omega t}$ and angular mode l are subject to a one dimensional scattering equation,

$$\frac{d^2\psi}{dr^{*2}} + \{\omega^2 - V(r)\}\psi = 0 \quad (5)$$

where

$$V(r) = \frac{\Delta}{r^4} \left(l(l+1) - \frac{6M}{r} \right) \quad (6)$$

The tortoise coordinate r^* , given as usual by $dr^* = r^2 dr/\Delta$, covers the range $-\infty < r^* < \infty$ as r ranges between the two horizons. The potential decays to zero exponentially fast in the r^* coordinate at both horizons. This is an important property which is not present for the black hole in an asymptotically flat spacetime and guarantees that the reflection and transmission coefficients are meromorphic functions of the frequency [21].

By definition, the quasinormal modes are ingoing waves at the event horizon and outgoing waves at the cosmological horizon,

$$\psi \rightarrow \begin{cases} e^{-i\omega r^*} & \text{as } r^* \rightarrow \infty \\ e^{i\omega r^*} & \text{as } r^* \rightarrow -\infty \end{cases} \quad (7)$$

The method which we use to determine the complex frequencies follows the continued fraction approach devised by Leaver [22]. The idea is to first scale out the divergent behavior at the cosmological horizon and examine the resulting Fröbenius series. If we begin with a purely divergent solution at the event horizon, the values of ω can be chosen to select the convergent rescaled solution at the cosmological horizon.

The analysis is slightly easier if we use the independent variable $x = 1/r$. In de Sitter space, the radial equation has three regular singular points x_1 (the event horizon), x_2 (the cosmological horizon) and x_3 . The indices of the equation at these points are $\pm\rho_i = \pm i\omega/(2\kappa_i)$, where

$$\kappa_1 = 2M(x_1 - x_2)(x_1 - x_3) \quad (8)$$

$$\kappa_2 = 2M(x_2 - x_3)(x_2 - x_1) \quad (9)$$

$$\kappa_3 = 2M(x_3 - x_1)(x_3 - x_2) \quad (10)$$

Note that $\rho_1 + \rho_2 + \rho_3 = 0$ and

$$e^{-i\omega r^*} = (x - x_1)^{-\rho_1} (x - x_2)^{-\rho_2} (x - x_3)^{-\rho_3} \quad (11)$$

We define,

$$\psi = e^{-i\omega r^*} u \quad (12)$$

For a quasinormal mode, $u(x)$ must be divergent at x_1 and finite at x_2 . The Frobenius series is therefore

$$u(x) = (x - x_1)^{2\rho_1} \sum_{n=1} a_n \left(\frac{x - x_1}{x_2 - x_1} \right)^n \quad (13)$$

where $a_0 = 1$ and the coefficients satisfy a recurrence relation of the form

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0. \quad (14)$$

The existence of this recurrence relation with only three terms is very important, because the convergence of the Fröbenius series at $x = x_2$ is now dependent on a continued fraction equation,

$$0 = \beta_0 - \frac{\alpha_0 \gamma_1}{\beta_1 -} \frac{\alpha_1 \gamma_2}{\beta_2 -} \dots \quad (15)$$

which can be solved very quickly numerically.

Figures 1 and 2 show the real and imaginary parts of the frequency of the $l = 2$ and $l = 3$ modes as $r_1 \kappa_1$ is varied. The quasinormal modes in flat space are recovered at $r_1 \kappa_1 = 0.5$.

The simple shape of the potential suggests that a Pöschl-Teller approximation might be effective, as it is for the Schwarzschild black hole [23]. The approximate potential

$$V_{PT} = V_0 \operatorname{sech}^2(r^*/b) \quad (16)$$

contains two free parameters which are used to fit the height and the second derivative of the potential at the maximum. The location of the maximum has to be found numerically.

The quasinormal modes of the Pöschl-Teller potential can be evaluated analytically,

$$\omega_n = \frac{1}{b} \left\{ (n + 1/2)i + \sqrt{b^2 V_0 - 1/4} \right\}. \quad (17)$$

for $n = 1, 2, \dots$. These values are compared to the numerical values for $l = 2$ in figure 1 and $l = 3$ in figure 2. The approximation works best for the imaginary parts of the frequencies, but the differences between the numerical values and the Pöschl-Teller values decrease as l is increased and soon become smaller than the resolution of such graphs.

For large l it is possible to obtain the Pöschl-Teller fit to the potential analytically. The maximum of the potential in this case lies close to $r = 3M$. From the second derivative at $r = 3M$, we get

$$b^{-1} = \kappa_1 (1 + \kappa_1 r_1)^{-1} (1 + \frac{2}{3} \kappa_1 r_1)^{1/2} \quad (18)$$

and $b^2 V_0 = l(l + 1)$. Equations (3) and (4) have been used to express the result in terms of the surface gravity. The analytic expression for the frequencies is therefore

$$\omega_n = \kappa_1 \frac{(1 + \frac{2}{3} \kappa_1 r_1)^{1/2}}{1 + \kappa_1 r_1} \left\{ (n + 1/2)i + (l(l + 1) - 1/4)^{1/2} \right\}. \quad (19)$$

Figures 1 and 2 show the numerical results and the approximation for small l . The difference is largest in the flat space limit at the right hand side of the figures. As l is increased, the difference between the approximation and the numerical results soon becomes insignificant.

III. ANTI-DE SITTER SPACE

The metric for a black hole in anti-de Sitter space takes the same form as for de Sitter space but with Λ negative. The cosmological horizon is no longer present. We introduce an anti-de Sitter radius a by the definition $a^2 = -3/\Lambda$. For fixed values of κ_1 and a there are two black hole solutions with different radii,

$$\kappa_1 r_1 = \frac{1}{3} (\kappa_1 a)^2 \pm \frac{1}{3} \left((\kappa_1 a)^4 - 3(\kappa_1 a)^2 \right)^{1/2} \quad (20)$$

The larger solution has positive and the smaller one negative specific heat.

The axial metric perturbations are subject to the same radial equation as for de Sitter space, however in anti-de Sitter space the tortoise coordinate lies in the range $-\infty < r^* \leq 0$. The boundary conditions at $r^* = 0$ can be dirichlet, neumann or robin depending respectively on whether the field, its derivative or a combination of both vanishes. Previous work has tended to opt for dirichlet boundary conditions [12], based largely on ADS invariance scalar field theory [24,25]. However, dirichlet boundary conditions break a duality in the equations between axial and polar metric perturbations [18]. We will introduce a set of self-dual boundary conditions based on the ADS-CFT correspondence.

In the Newman-Penrose formalism, the gravitational perturbations can be described by two Weyl scalars Ψ_0 and Ψ_4 . In flat space, Ψ_0 would represent gravitational waves which are incoming at infinity and Ψ_4 would represent outgoing gravitational waves. The equations

for the Weyl scalars separate to give a generalisation of the Teukolski equations [15]. We put, for example,

$$\Psi_0 = e^{i\omega t} \frac{r^3}{\Delta} Y(r) S_l(\theta) \quad (21)$$

where $S_l(\theta)$ can be found in Chandrasekhar [18]. The radial equation for $Y(r)$ can be related to the radial equation for the metric perturbations (5) if we set

$$Y = V^\pm \psi^\pm + (W^\pm + 2i\omega) \Lambda_+ \psi^\pm \quad (22)$$

where

$$\Lambda_+ = \frac{d}{dr^*} + i\omega. \quad (23)$$

There are two equivalent formulations corresponding to polar and axial metric perturbations, denoted by the plus and minus signs. The axial form, corresponding to ψ^- , has $V^-(r) = V(r)$ given previously (6).

The axial and polar functions are related by

$$C^\mp \psi^\mp = (C^\mp + 72M^2 f) \psi^\pm \mp 12M \Lambda_- \psi^\pm \quad (24)$$

with $C^\pm = \mu^2(\mu^2 + 2) \pm 12iM\omega$, and

$$f = \frac{\Delta}{r^3} \frac{1}{\mu^2 r + 6M}. \quad (25)$$

where $\mu^2 = (l-1)(l+2)$.

The ADS-CFT correspondence conjectures that when the Anti-de Sitter space is conformally compactified, the generating functions of the conformal field theory on the boundary are related to the partition function of the superstring theory on the interior, or

$$\left\langle \exp \left(\int \hat{h}_{\mu\nu} T^{\mu\nu} \right) \right\rangle_{CFT} = Z(\hat{h}_{\mu\nu}) \quad (26)$$

If we calculate the correlation functions of a polar quantity, such as the energy, we must therefore set the axial metric perturbation to vanish on the boundary. The transformation theory (24) then implies a Robin boundary condition on the polar metric perturbation, i.e. the normal derivative is determined by the value of the field. The same set of quasinormal modes is obtained from the axial and polar equations and the duality between the two is preserved.

In general, both the axial and polar metric variations are non-vanishing. This is consistent with the conformal theory viewpoint, where the six components of the metric perturbations in three dimensions are subject to three coordinate transformations and the conformal symmetry, leaving two independent functions. If we define the ratio $\gamma = \psi^+/\psi^-$, the two metric variations are related by (24), which gives a robin boundary condition

$$(\xi + \Lambda_-) \psi^- \quad (27)$$

where

$$\xi = (1 - \gamma) \frac{C^+}{12M} + \frac{6M}{\mu^2 a^2} \quad (28)$$

Some special solutions lead to vanishing of the Newmann-Penrose scalars and might be regarded as pure gauge modes from the point of view of the ADS theory. For example,

$$\psi^- = e^{-i\omega r^*} \left(1 + \frac{6M}{\mu^2 r} \right) \quad (29)$$

with $\omega = -\mu^2(\mu^2 + 2)i/12M$, obtained by setting $\psi^+ = 0$, is a purely axial perturbation with $Y = 0$. Replacing ω by $-\omega$ gives a quasinormal mode with frequency

$$\omega = \frac{\mu^2(\mu^2 + 2)}{12M} i \quad (30)$$

Such modes appear in the numerical solutions discussed below.

The frequencies of the quasinormal modes can be found numerically by adapting the method used for de Sitter space. We first scale out the behaviour of the quasinormal mode at the event horizon,

$$\psi^- = e^{i\omega r^*} u, \quad (31)$$

where u will be regarded as a function of $x = 1/r$. The radial equation (5) gives the equation

$$p u'' + p' u' + 2i\omega u' + \{l(l+1) - 6Mx\} u = 0 \quad (32)$$

where $p = -a^{-2} - x^2 + 2Mx^3$. We take the Fröbenius series which is regular at the event horizon,

$$u = \sum_{n=0}^{\infty} a_n \left(\frac{x - x_1}{-x_1} \right)^n \quad (33)$$

with $a_0 = 1$. The coefficients are obtained from a recurrence relation similar to (14). The series can then be substituted into the boundary condition $u = 0$ at $x = 0$ to obtain an equation for the frequencies of the quasinormal modes with vanishing axial perturbations.

The frequencies of the first few modes using dirichlet boundary conditions on the axial perturbations have been plotted for $a\kappa_1 = 2.0$ and $r_1 = a$ in figure 3. For $l = 2$, one of the frequencies lies on the imaginary axis. (This is also found by Cardoso et al [11]). We have found a single frequency on the imaginary axis and can trace it for $\kappa_1 r_1 \geq 1.3$ when $l = 2$, for $\kappa_1 r_1 \geq 3.4$ when $l = 3$ and for $\kappa_1 r_1 \geq 38.0$ when $l = 10$.

The dependence of the first three modes on the metric parameters, with $l = 2$ and $l = 10$, is displayed in figures 3 and 4. Both the real and imaginary parts of the frequencies become proportional to κ_1 for large κ_1 , which agrees with the behaviour for the scalar field quasinormal modes [10]. For $r_1 \ll a$, the numerical data suggests that imaginary part of the frequency ω_I approaches zero as in the scalar case [10].

Corresponding results for dirichlet boundary conditions on the polar perturbations ($\gamma = 0$) are shown in figures 5 and 6. The special mode in this case is the ‘pure gauge’ mode (30)

and the analytic solution provides a good check on the accuracy of our numerical technique. The frequencies for these boundary conditions are not drastically different from the dirichlet axial boundary conditions. The two sets of results represent two extreme cases of the one parameter family of boundary conditions (27) given above, and indicate a weak dependence on the parameter γ .

The potential $V(r)$ for anti-de Sitter space no longer vanishes as $r \rightarrow \infty$ and it is harder to fit the potential to a Pöschl-Teller form. More importantly, the value of r^* at infinity is zero, and the asymptotic forms of the mode functions for the Pöschl-Teller potential can no longer be used.

IV. CONCLUSION

The quasinormal modes for the gravitational perturbations of a black hole in de Sitter space have much in common with the quasinormal modes for the black hole in asymptotically flat spacetime. The radial scattering problem in de Sitter space is better behaved because the potential falls off exponentially rather than as a power law. This explains why the approximation to the quasinormal mode frequencies, derived in section 2 from a Pöschl-Teller potential, is in such good agreement with the numerical results for even moderate angular order l .

The corresponding problem in anti-de Sitter space requires careful consideration of the boundary conditions at infinity. The proposal here has been to use the ADS-CFT correspondence and the duality symmetry between axial and polar metric perturbations. For correlation functions of the energy in the conformal field theory, the appropriate boundary conditions are that the axial metric perturbations vanish and the polar perturbations satisfy a robin boundary condition.

In general, two functions have to be specified on the boundary and there is a one parameter family of boundary conditions. The effect on the quasinormal frequencies is quite small.

It would be desirable to analyse thermal widths in three dimensional conformal field theory to look for signs of a correspondence with the quasinormal mode frequencies found here. This could give an impetus for analysing the five dimensional gravitational quasinormal modes, where a full separation of the gravitational perturbation equations is still an unsolved problem.

FIGURES

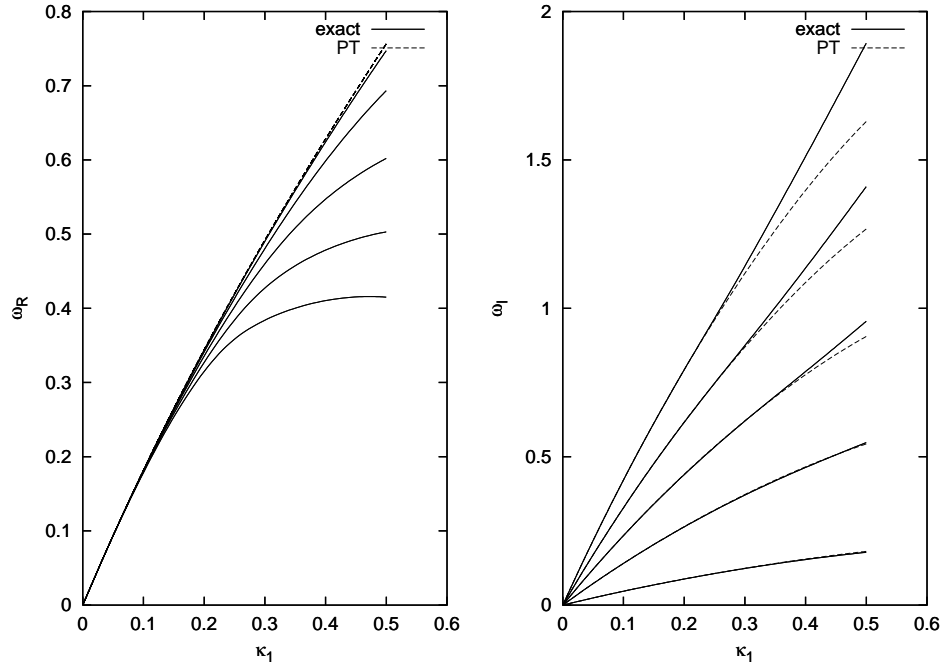


FIG. 1. Real (left) and imaginary (right) parts of the gravitational quasinormal mode frequencies $r_1\omega$ with $l = 2$ plotted as a function of the surface gravity $r_1\kappa_1$, together with the Pöschl-Teller approximation.

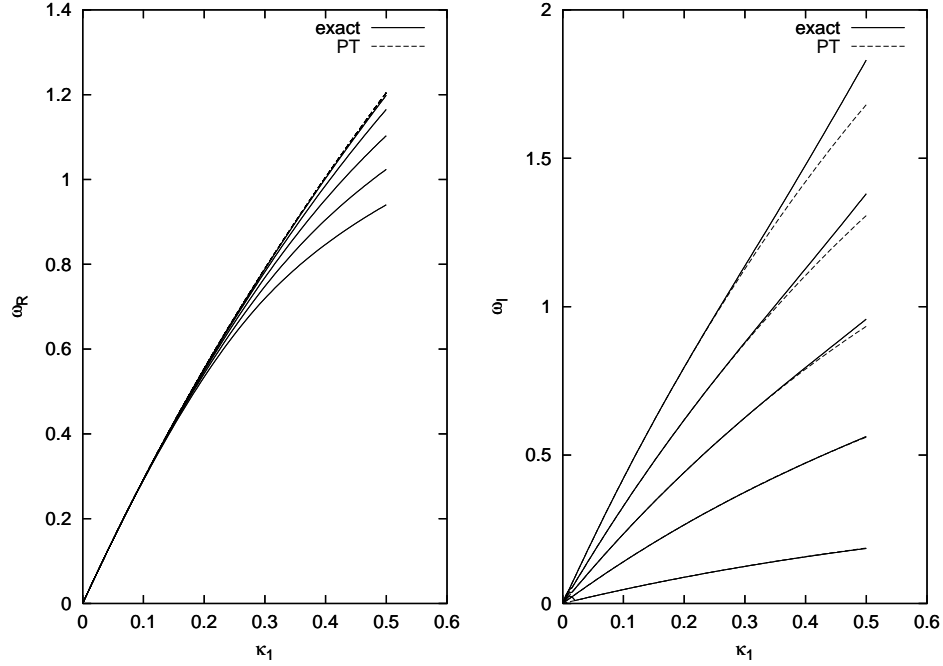


FIG. 2. Real (left) and imaginary (right) parts of the gravitational quasinormal mode frequencies $r_1\omega$ with $l = 3$ plotted as a function of the surface gravity $r_1\kappa_1$, together with the Pöschl-Teller approximation.

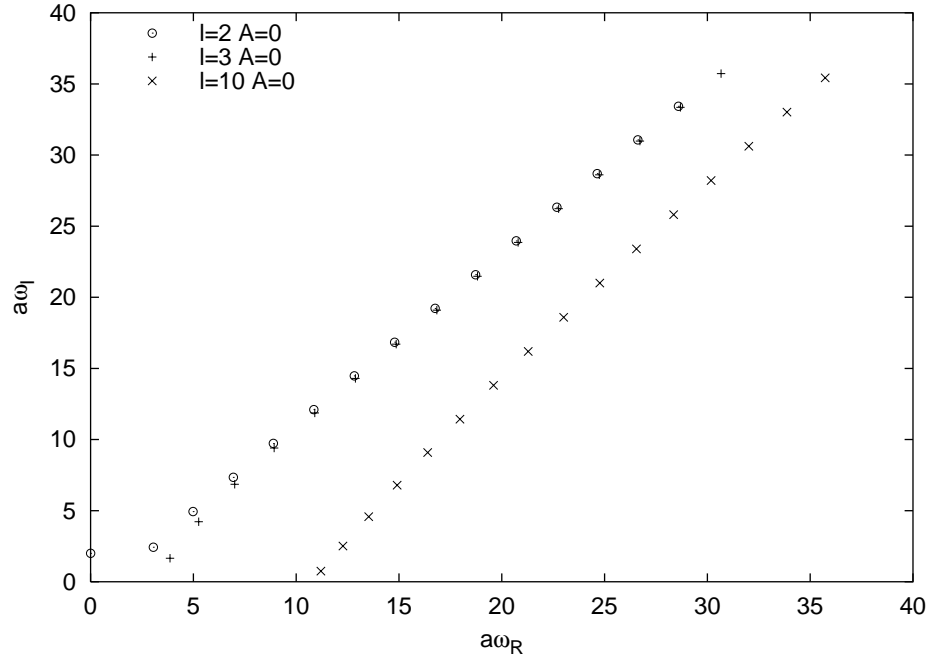


FIG. 3. The frequencies of the first few quasinormal modes for the black hole anti-de Sitter metric with dirichlet axial boundary conditions and $a\kappa_1 = 2.0$ (and $r_1 = a$). Results for $l = 2, 3$ and 10 are shown.

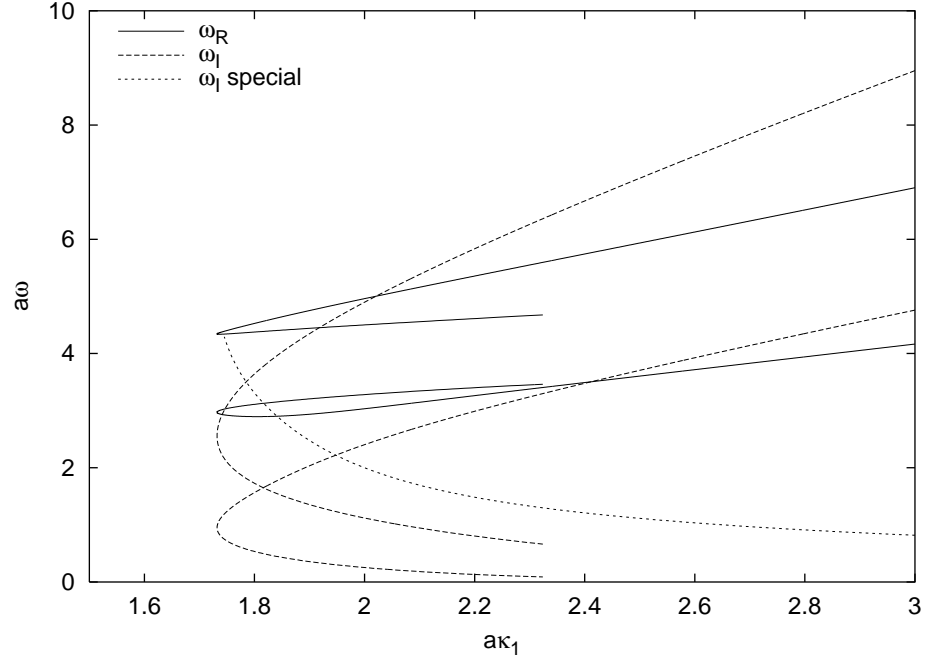


FIG. 4. The dependence of the first two quasinormal mode frequencies on the metric parameters for $l = 2$ is shown for the dirichlet axial boundary conditions. The frequency with $\omega_R = 0$ is also plotted and designated 'special'.

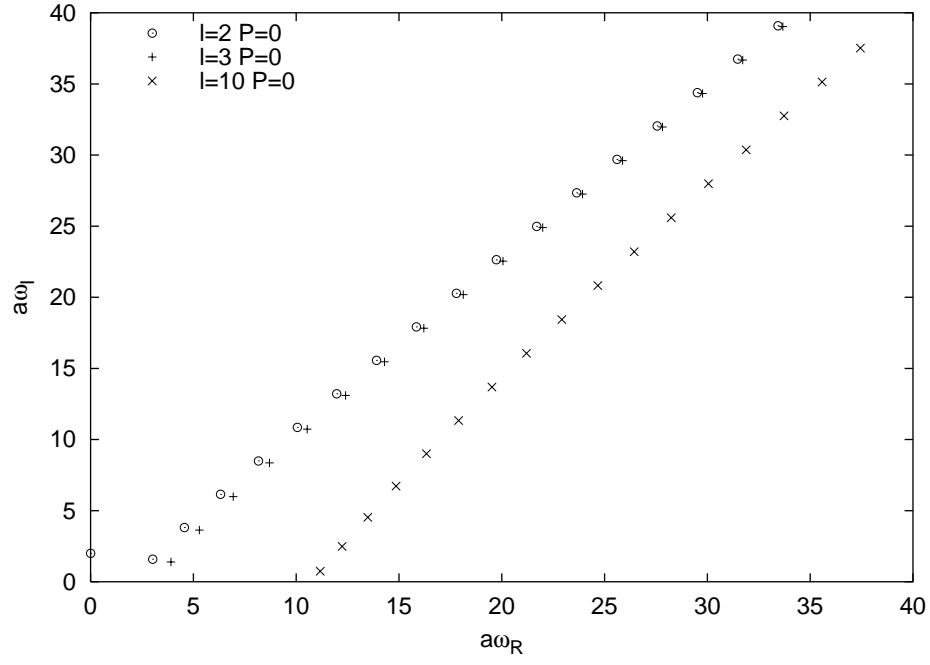


FIG. 5. The frequencies of the first few quasinormal modes for the black hole anti-de Sitter metric with the dirichlet polar boundary conditions ($\gamma = 0$) and $\kappa_1 a = 2.0$ (and $r_1 = a$). Results for $l = 2, 3$ and 10 are shown.

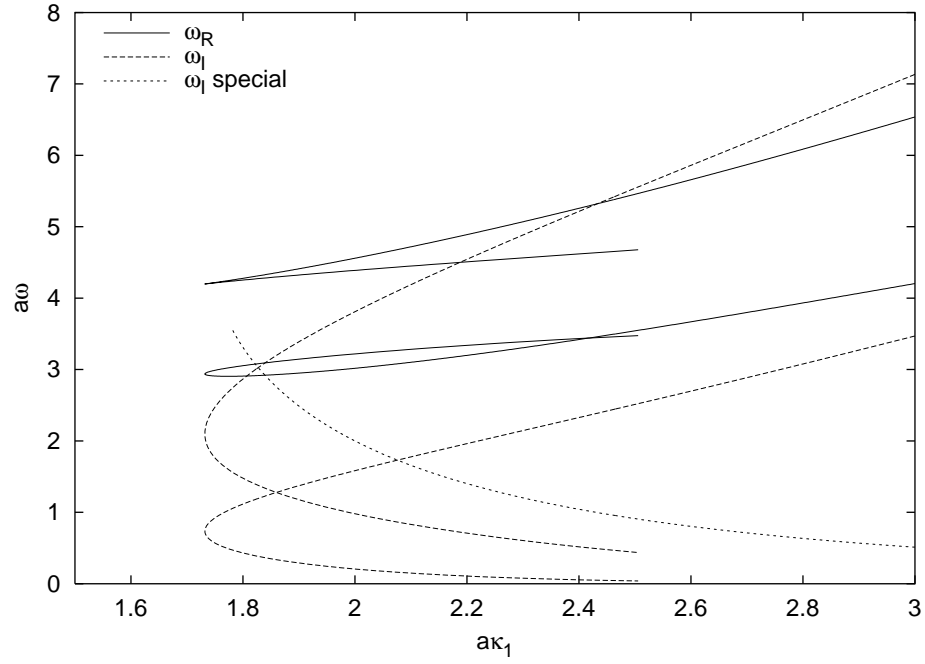


FIG. 6. The dependence of the first two quasinormal mode frequencies on the metric parameters for $l = 2$ is shown for the dirichlet polar boundary conditions ($\gamma = 0$). The frequency with $\omega_R = 0$ is also plotted and designated ‘special’.

REFERENCES

- [1] Nollert H P 2000 *Class Quantum Grav.* **1**
- [2] Kokkotas K D and Schmidt B G 1999 *Living Rev. Rel.* **2** (www.livingreviews.org)
- [3] Maldacina J 1998 *Adv. Theor. Math. Phys* **2** 231
- [4] Witten E 1998 *Adv. Theor. Math. Phys.* **2** 253
- [5] Witten E 1998 *Adv. Theor. Math. Phys.* **2**
- [6] Hawking S W and Page D 1983 **87** 577
- [7] Chan J S F and Mann R B 1997, *Phys. Rev. D* **55**, 7546
- [8] Chan J S F and Mann R B 1999, *Phys. Rev. D* **59**, 064025
- [9] Horowitz G T 2000 *Class. Quantum Grav.* **17** 1107
- [10] Horowitz G T and Hubeny V 2000 *Phys. Rev. D* **62** 024027
- [11] Cardoso V and Lemos J P S 2001 *Phys. Rev. D* **63** 124015
- [12] Cardoso V and Lemos J P S 2001 *gr-qc/0105103*
- [13] Mellor F and Moss I G 1990 *Phys. Rev. D* **41** 4031992; 1992 *Class. Quantum Grav.* **9** L43
- [14] Guven G 1990 *Phys. Rev. D* **42** 2577
- [15] Chambers C G and Moss I G 1993 *Class. Quantum Grav.* **11** 1035
- [16] Moss I G 2002 *Nucl. Phys. B (Pro. Suppl.)* **104** 181
- [17] Regge T and Wheeler J A 1957 *Phys. Rev.* **108** 1063
- [18] Chandrasekhar S 1983 *The mathematical theory of black holes* (Cambridge University Press, UK)
- [19] Chandrasekhar S and Hartle J B 1982 *Proc. R. Soc.* **484** 301
- [20] Brady P R, Moss I G and Myers R 1998 *Phys. Rev. Lett.* **80** 3432
- [21] Newton R G 1966 *Scattering theory of waves and particles* (McGraw-Hill, New York)
- [22] Leaver E W 1985 *Proc. Roy. Soc. Lon.* 285
- [23] Ferrari V and Mashhoon B 1984 *Phys. Rev. Lett.* **52** 1361
- [24] Avis S J, Isham C J and Storey D 1978 *Phys. Rev. D* **18** 3565
- [25] Breitenlohner and Freedman D Z 1982 *Phys. Lett.* **B153** 137